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LETTER TO THE EDITOR

A solution of the generalised non-linear Schrödinger equation

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Abstract. We first present a non-linear Schrödinger equation that describes wave propagation in fluids and plasmas with sharp boundaries and dissipation. Then we show that an exact solution can be found.

In fluids and plasmas one often (e.g. [1]) considers the propagation of a large-amplitude plane wave $\psi\exp(\mathrm{i}k_0x-\mathrm{i}\omega_0t)$, where ψ is a slowly varying complex envelope function of the modulated wave with carrier frequency ω_0 and wavenumber k_0, and derives a dispersion relation

$$\omega=\omega(k,|\psi|^2). \tag{1}$$

Equation (1) is thereafter used to construct a wave equation by the expansion

$$\omega-\omega_0=(k-k_0)\frac{\partial\omega}{\partial k_0}+[(k-k_0)^2/2]\frac{\partial^2\omega}{\partial k_0^2}+|\psi|^2\frac{\partial\omega}{\partial|\psi|^2}.$$

One then substitutes the operator $\mathrm{i}\,\partial/\partial t$ for $\omega-\omega_0$ and $-\mathrm{i}\,\partial/\partial x$ for $k-k_0$. However, for $(k-k_0)^2$ one may substitute either $-\partial^2/\partial x^2$ or $-(\partial/\partial x)^2$. Transforming to a coordinate system moving at the group speed, i.e. $x\to x-(\partial\omega/\partial k_0)t$, one thus obtains the equation

$$\mathrm{i}\frac{\partial\psi}{\partial t}+p_0\frac{\partial^2\psi}{\partial x^2}+q_0|\psi|\psi+C_0\left[\frac{\partial^2\psi}{\partial x^2}-\frac{1}{\psi}\left(\frac{\partial\psi}{\partial x}\right)^2\right]=0 \tag{2}$$

where $p_0=\frac{1}{2}\partial^2\omega_0/\partial k_0^2$ and $q_0=-\partial\omega_0/\partial|\psi|^2$, and where C_0 is an arbitrary constant.

Equation (2) has previously been found by Gradov and Stenflo [2] in their studies of plasmas with sharp boundaries. It should also be stressed that the corresponding textbook derivations (e.g. [1]) of (2) all overlook the C_0 term.

Here we shall generalise the Gradov–Stenflo equation (2) to allow the coefficients to be complex. Thus we introduce the equation

$$\mathrm{i}\,\partial_t\psi+(p+C)\,\partial_x^2\psi+q|\psi|^2\psi=C(\partial_x\psi)^2/\psi+\mathrm{i}\gamma\psi \tag{3}$$

where p, q, γ and C are arbitrary complex parameters, which we write as $p=p_\mathrm{r}+\mathrm{i}p_\mathrm{i}$, etc. In the particular case where C is zero, we note that (3) reduces to the well known Ginzburg–Landau equation with complex coefficients (e.g. [3–12]). That equation has a wide range of applications, as, for example (see [6] and the references therein), for phase transitions in non-equilibrium systems, Bénard convection, Taylor–Couette flow, Poiseuille flow in fluid systems, drift dissipative waves in plasma physics, chemical turbulence and ionisation waves in glow discharges. However, in the above-mentioned examples one has to introduce (3), with non-zero C_r and C_i, if sharp boundaries are present [2].

The solutions of (3), which may describe a chaotic state, are of course very complex. Here we shall just point out that, despite the complexity of (3), it is possible to find an exact and comparatively very simple solution of that equation. Thus, as can easily be verified by direct substitution, the function

$$\psi(x, t) = \psi_0(\cosh Kx)^{-1-i\alpha} \exp(-i\Omega t) \tag{4}$$

satisfies (3) if

$$\alpha = -\beta \pm \left(2 + \beta^2 + \frac{C_r q_i - q_r C_i}{p_r q_i - q_r p_i}\right)^{1/2}$$

where

$$\beta = \frac{3(p_r q_r + p_i q_i) + q_r C_r + q_i C_i}{2(p_r q_i - q_r p_i)}$$

$$K = [\gamma_r/(2\alpha p_r + p_i - \alpha^2 p_i)]^{1/2}$$

$$\Omega = -\gamma_i - \gamma_r(p_r - \alpha^2 p_r - 2\alpha p_i)/(2\alpha p_r + p_i - \alpha^2 p_i)$$

and

$$\psi_0 = \left(\frac{\gamma_r(2p_r - \alpha^2 p_r - 3\alpha p_i + C_r - \alpha C_i)}{q_r(2\alpha p_r + p_i - \alpha^2 p_i)}\right)^{1/2}.$$

The present approach represents evidently only a first step in the investigation of (3). Accordingly, at this stage of the analysis we cannot know whether the solution (4) is stable or unstable. In forthcoming studies it should therefore be of interest to generalise (3) to the three-dimensional case, for example by adding a $\partial^2\psi/\partial y^2$ term (see, e.g., [5, 12]) to (3).

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